Discrete Mathematics 223 (2000) 287–297

DISCRETE MATHEMATICS

On the number of large h-hypergraphs with a fixed diameter

Ioan Tomescu

Faculty of Mathematics, University of Bucharest, Str. Academiei, 14, 70109 Bucharest, Romania

Received 4 November 1998; revised 23 November 1998; accepted 19 November 1999

Abstract

An asymptotic formula for the number of labeled h-hypergraphs of order n and diameter equal to k, denoted by $H(n,h;d=k)$ is found as $n\to\infty$ for every $h\geqslant 3$ and $k\geqslant 2$. This depends on the parity of k and is more precise for $k\geqslant 5$. As a consequence, it is deduced that for every fixed $h\geqslant 3$ and $k\geqslant 1$, $\lim_{n\to\infty} H(n,h;d=k)/H(n,h;d=k+1)=\infty$ holds, thus extending the corresponding results for graphs and digraphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Definitions and notation

A simple hypergraph $H=(X,\mathscr{E})$, with order $n=|X|$ and size $m=|\mathscr{E}|$, consists of a vertex-set $V(H)=X$ and an edge-set $E(H)=\mathscr{E}$, where $E\subseteq X$ and $|E|\geqslant 2$ for each edge E in $\mathscr{E}$. H is h-uniform, or is an h-hypergraph, if $|E|=h$ for each E in $\mathscr{E}$. The number of edges containing a vertex x is its degree $d_H(x)$.

Two vertices u,v of H are in the same component if there are vertices $x_0=u,x_1,\dots,x_k=v$ and edges $E_1,\dots,E_k$ of H such that $x_{i-1},x_i\in E_i$ for each $i(1\leqslant i\leqslant k)$. If H has only one component then it is connected. A path P of length k in H [1] is a subhypergraph comprising $k+1$ distinct vertices $x_1,\dots,x_{k+1}$ and k distinct edges $E_1,\dots,E_k$ of H such that $x_i,x_{i+1}\in E_i$ for each i, $1\leqslant i\leqslant k$. For a connected hypergraph H the distance $d(x,y)$ between vertices x and y is the length of a shortest path between them. The eccentricity of a vertex x is $\mathrm{ecc}(x)=\max_{y\in V(H)} d(x,y)$. The diameter of H, denoted $d(H)$, is equal to $\max_{x\in V(H)}\mathrm{ecc}(x)=\max_{x,y\in V(H)} d(x,y)$ if H is connected and ∞ otherwise. Let $A_{ij}^{(k)}$ be the set of h-hypergraphs H with vertex set $\{1,\dots,n\}$ such that the distance between vertices i and j is at least k. By $H(n,h;d=k)$ and $H(n,h;d\geqslant k)$ we denote the number of labeled h-hypergraphs H of order n and diameter $d(H)=k$ and $d(H)\geqslant k$, respectively.

E-mail address: ioan@colegiu.cs.unibuc.ro (I. Tomescu).

PII: S0012-365X(00)00019-4

We say that almost all h-hypergraphs have a property P when the ratio between the number of labeled h-hypergraphs of order n possessing P and the number of all labeled h-hypergraphs of order n, i.e. $2^{\binom{n}{h}}$, approaches 1 as $n \to \infty$. For every real x and natural $n \geqslant 1$ let $(x)_n$ denote the product $x(x-1)\dots(x-n+1)$.

2. Preliminary results

Contrary to the case of graphs (when almost all graphs have diameter two [2]), almost all h-hypergraphs have diameter equal to one for every $h \geqslant 3$:

Lemma 2.1. *For every $h \geqslant 3$ almost all h-hypergraphs H of order n have diameter one as $n \to \infty$.*

Proof. Since $A_{ij}^{(2)}$ for $1 \leqslant i < j \leqslant n$ is the set of h-hypergraphs such that vertices i and j are not contained in any edge of H it follows that $|A_{ij}^{(2)}| = 2^{\binom{n}{h}-\binom{n-2}{h-2}}$. The set of h-hypergraphs of order n with diameter greater than one is precisely $\bigcup_{1 \leqslant i<j \leqslant n} A_{ij}^{(2)}$ and

$$\left| \bigcup_{1 \leqslant i<j \leqslant n} A_{ij}^{(2)} \right| \leqslant \sum_{1 \leqslant i<j \leqslant n} |A_{ij}^{(2)}| = \binom{n}{2} 2^{\binom{n}{h}-\binom{n-2}{h-2}}, \tag{1}$$

hence $\lim_{n\to\infty} |\bigcup_{1 \leqslant i<j \leqslant n} A_{ij}^{(2)}| / 2^{\binom{n}{h}} = 0$ for every $h \geqslant 3$. $\square$

Lemma 2.2. *Let A, B be two disjoint sets such that $|A| = p$ and $|B| = q$. Then the number of h-hypergraphs H with vertex set $V(H) = A \cup B$, $E(H)$ has no edge included in A or in B and $d_H(x) \geqslant 1$ for every vertex $x \in B$ equals*

$$a(p,q) = \alpha(p,q) - \binom{q}{1}\alpha(p,q-1) + \binom{q}{2}\alpha(p,q-2) - \cdots + (-1)^q, \tag{2}$$

where $\alpha(p,q) = 2^{\binom{p+q}{h}-\binom{p}{h}-\binom{q}{h}}$ for every $p, q \geqslant 0$.

Proof. Note that $\alpha(p,q)$ is the number of h-hypergraphs H such that $V(H) = A \cup B$ and no edge of H is included in A or in B. Now (2) follows by the inclusion-exclusion principle. $\square$

It is clear that $a(p,q) = 1$ for every $p, q \geqslant 1$ such that $p + q = h$.

Lemma 2.3. *If $q \geqslant 1$ is fixed and $p \to \infty$ or if $p \geqslant 1$ is fixed and $q \to \infty$ then*

$$a(p,q) = \alpha(p,q)(1 + o(1))$$

holds.

Proof. The property follows from the Bonferroni inequalities

$$\alpha(p,q) - \binom{q}{1}\alpha(p,q-1) \leqslant a(p,q) \leqslant \alpha(p,q)$$

since $h \geqslant 3$. □

In the next chapter we need the following extremal property of an arithmetic function. Let $k \geqslant 3$ and

$$g(n,h;n_1,\ldots,n_k) = \binom{n}{n_1,\ldots,n_k} 2^{\sum_{i=1}^{k-1}\binom{n_i+n_{i+1}}{h} - \sum_{i=2}^{k-1}\binom{n_i}{h} - |\{i: n_i+n_{i+1}=h\}|}, \qquad (3)$$

where $(n_1,\ldots,n_k)$ belongs to the domain $D(n,k,h)$ defined by: $n_1,\ldots,n_k$ are positive integers such that $n_1+\cdots+n_k=n$;

$$n_1 = 1 \text{ and } n_i + n_{i+1} \geqslant h \text{ for every } i = 1,\ldots,k-1. \qquad (4)$$

We denote

$$g(n,k,h) = \max_{(n_1,\ldots,n_k)\in D(n,k,h)} g(n,h;n_1,\ldots,n_k).$$

Note that the function $g(n,h;n_1,\ldots,n_k)$ counts the number of labeled h-hypergraphs with a certain layered structure, as we shall see later.

Theorem 2.4. *We have* $g(n,3,h) = n(n-1)2^{\binom{n}{h}-\binom{n-2}{h-2}} = n(n-1)2^{\binom{n}{h}-[1/(h-2)!]n^{h-2}+O(n^{h-3})}$ *for every* $h \geqslant 3$ *and*

$$g(n,k,h) = \frac{(n)_{n-(k-1)h/2}}{((h-1)!)^{(k-1)/2}} 2^{2\binom{n-(k-1)h/2+h-1}{h} - \binom{n-(k-1)h/2}{h}}$$

$$= 2^{\binom{n}{h} + n^{h-1}\beta(k,h) + O(n^{h-2})} \quad \text{for every fixed } h \geqslant 3 \text{ and odd } k \geqslant 5;$$

$$g(n,k,h) = \frac{(n)_{n-kh/2+h-1}}{((h-1)!)^{(k-2)/2}} 2^{\binom{n-kh/2+2h-2}{h} + \binom{n-kh/2+h-1}{h-1}}$$

$$= 2^{\binom{n}{h} + n^{h-1}\beta(k,h) + O(n^{h-2})} \quad \text{for every fixed } h \geqslant 3 \text{ and even } k \geqslant 4,$$

where

$$\beta(k,h) = \frac{1}{2(h-1)!}(h(5-k)-4) \quad \text{for odd } k \geqslant 5$$

and

$$\beta(k,h) = \frac{1}{2(h-1)!}(h(4-k)-2) \quad \text{for even } k \geqslant 4.$$

Proof. We will deduce some properties of the systems $(m_1,\ldots,m_k) \in D(n,k,h)$ that maximize $g(n,h;n_1,\ldots,n_k)$.

Claim 1. *We have* $m_k = 1$.

Let $\alpha \in \mathbb{N}$, $\alpha \geqslant 1$ and $n_k = \alpha + 1$; we deduce that

$$\frac{g(n,h;n_1,\ldots,n_{k-1}+1,\alpha)}{g(n,h;n_1,\ldots,n_{k-1},\alpha+1)} \geqslant \frac{\alpha+1}{n_{k-1}+1} 2^{\binom{n_{k-1}+n_{k-2}}{h-1}-\binom{n_{k-1}}{h-1}}.$$

If $n_{k-1} \leqslant h-1$, since $n_{k-1}+n_{k-2} \geqslant h$ this ratio is greater than or equal to $(2/h)2^{\binom{h}{h-1}-1} = 2^h/h > 1$ for $h \geqslant 3$; otherwise $n_{k-1} \geqslant h$ and this ratio is bounded below by

$$\frac{\alpha+1}{n_{k-1}+1} 2^{\binom{n_{k-1}+1}{h-1}-\binom{n_{k-1}}{h-1}} = \frac{\alpha+1}{n_{k-1}+1} 2^{\binom{n_{k-1}}{h-2}} \geqslant \frac{2^{n_{k-1}+1}}{n_{k-1}+1} > 1.$$

This shows that $m_k = 1$ for any k-tuple $(m_1,\ldots,m_k)$ maximizing $g(n,h;n_1,\ldots,n_k)$.

Claim 2. *If* $k \geqslant 4$ *then* $\min(m_i, m_{i+1}) \leqslant h-1$ *for every* $i = 2,\ldots,k-2$.

Suppose that $(n_1,\ldots,n_k) = (n_1,\ldots,\alpha,a,b,\beta,\ldots,n_k)$ where $a,b \geqslant h$. By denoting

$$R_1 = \frac{g(n,h;n_1,\ldots,\alpha,a-1,b+1,\beta,\ldots,n_k)}{g(n,h;n_1,\ldots,n_k)}$$

and

$$R_2 = \frac{g(n,h;n_1,\ldots,\alpha,a+1,b-1,\beta,\ldots,n_k)}{g(n,h;n_1,\ldots,n_k)}$$

we deduce that $R_1R_2 \geqslant [ab/(a+1)(b+1)]2^{f(a,b,\alpha,\beta,h)-1}$, where

$$f(a,b,\alpha,\beta,h) = \binom{b+\beta-1}{h-2} - \binom{b-1}{h-2} + \binom{a+\alpha-1}{h-2} - \binom{a-1}{h-2} \tag{5}$$

unless $a=b=h$ and $\alpha=\beta=1$, when $R_1R_2 = [ab/(a+1)(b+1)]2^{f(a,b,\alpha,\beta,h)-2}$. We get

$$f(a,b,\alpha,\beta,h) \geqslant \binom{b-1}{h-3} + \binom{a-1}{h-3} \geqslant 2$$

since $\alpha,\beta \geqslant 1$ and a, $b \geqslant h \geqslant 3$ and $f(a,b,\alpha,\beta,h) = 2$ only for $h=3$ and $\alpha=\beta=1$. It follows that for $h \geqslant 4$ or for $h=3$ and $\max(a,b) \geqslant 4$ or for $h=3$ and $\max(\alpha,\beta) \geqslant 2$ we have $R_1R_2 \geqslant [2ab/(a+1)(b+1)] > 1$. Consequently, in this case $R_1R_2 > 1$ thus implying $\max(R_1,R_2) > 1$ and $(n_1,\ldots,n_k)$ cannot be a k-tuple maximizing g.

It remains to consider the case when $h=3$, $a=b=3$ and $\alpha=\beta=1$. There exists a sufficiently large component n_p of $(n_1,\ldots,n_k)$ as $n \to \infty$. In this case

$$\frac{g(n,h;n_1,\ldots,1,2,3,1,\ldots,\gamma,n_p+1,\delta,\ldots,n_k)}{g(n,h;n_1,\ldots,1,3,3,1,\ldots,\gamma,n_p,\delta,\ldots,n_k)} > 1$$

and $(n_1,\ldots,n_k)$ cannot again maximize g.

Claim 3. *Let* a, $b \geqslant h$ *and* $1 \leqslant \alpha,\beta,\gamma,\delta \leqslant h-1$. *Then* $(n_1,\ldots,\alpha,a,\beta,\ldots,\gamma,b,\delta,\ldots,n_k)$ *cannot be a* k*-tuple maximizing* g.

If

$$R_3 = \frac{g(n,h;n_1,\dots,\alpha,a-1,\beta,\dots,\gamma,b+1,\delta,\dots,n_k)}{g(n,h;n_1,\dots,n_k)}$$

and

$$R_4 = \frac{g(n,h;n_1,\dots,\alpha,a+1,\beta,\dots,\gamma,b-1,\delta,\dots,n_k)}{g(n,h;n_1,\dots,n_k)}$$

then

$$R_3R_4 \geqslant \frac{ab}{(a+1)(b+1)} 2^{f(a,b,\alpha,\gamma,h)+\binom{a+\beta-1}{h-2}+\binom{b+\delta-1}{h-2}-4} > \frac{4ab}{(a+1)(b+1)} > 1,$$

where f is defined by (5).

In a similar way we can prove the following two claims:

Claim 4. *Let $a,b \geqslant h$ and $1 \leqslant \alpha,\beta,\gamma \leqslant h-1$. Then the k-tuple $(n_1,\dots,\alpha,a,\beta,b,\gamma,\dots,n_k)$ cannot maximize g.*

It follows that in the sequence $(m_1,\dots,m_k)$ exactly one member is greater than $h-1$ for sufficiently large n.

Claim 5. *For $k \geqslant 4$ if the k-tuple $(m_1,\dots,m_k)$ maximizes g then for sufficiently large n we have $\max(m_i,m_{i+1}) \leqslant h-1$ implies $m_i+m_{i+1}=h$ for every $1 \leqslant i \leqslant k-1$.*

Let $(m_1,\dots,m_k)=(\dots,x,y,z,\dots)$, where $1 \leqslant x,z \leqslant h-1$ and $y \geqslant h$. Then it is clear that there exist two constants C_1, $C_2>0$ (depending only on h and k) such that

$$C_1\varphi(x,y,z) \leqslant \frac{g(n,h;m_1,\dots,m_k)}{n!} \leqslant C_2\varphi(x,y,z) \tag{6}$$

as $n \to \infty$, where

$$\varphi(x,y,z) = \frac{1}{y!} 2^{\binom{x+y}{h}+\binom{y+z}{h}-\binom{y}{h}} \quad \text{and} \quad \varphi(x,y,z)=\varphi(z,y,x).$$

One deduces that

$$\begin{aligned} \frac{\varphi(x,y+1,z)}{\varphi(x,y,z)} &= \frac{1}{y+1} 2^{\binom{x+y}{h-1}+\binom{y+z}{h-1}-\binom{y}{h-1}} \\ &\geqslant \frac{1}{y+1} 2^{\binom{y+1}{h-1}+\binom{y}{h-2}} \to \infty \quad \text{as } y \to \infty \\ &\text{(hence as } n \to \infty). \end{aligned}$$

Also

$$\begin{aligned} \frac{\varphi(x-1,y+1,z)}{\varphi(x,y,z)} &= \frac{x}{y+1} 2^{\binom{y+z}{h-1}-\binom{y}{h-1}} \\ &\geqslant \frac{x}{y+1} 2^{\binom{y}{h-2}} \to \infty \quad \text{as } y \to \infty. \end{aligned}$$

From (6) it follows that the single member y greater than $h-1$ in the sequence $(m_1,\dots,m_k)$ must have a maximal value in the domain $D(n,k,h)$ for sufficiently large n. We shall consider two cases: I. k is odd; II. k is even.

I. Let k odd, $k\geqslant 5$ and $y=m_r$, where $2\leqslant r\leqslant k-1$. If r is odd, by summing up the inequalities $m_1+m_2\geqslant h,\dots,m_{r-2}+m_{r-1}\geqslant h$, $m_{r+1}+m_{r+2}\geqslant h,\dots,m_{k-1}+m_k\geqslant h$ one gets $y\leqslant n-(k-1)h/2$. Because $m_1=m_k=1$ and $m_i+m_{i+1}\geqslant h$ for every $1\leqslant i\leqslant k-1$ we deduce that $y=n-(k-1)h/2$ for $1\leqslant x\leqslant h-1$ and $1\leqslant z\leqslant h-1$. But

$$\frac{\varphi(x+1,y,z)}{\varphi(x,y,z)}=\frac{1}{x+1}2^{\binom{x+y}{h-1}}\to\infty \quad \text{as } y\to\infty.$$

A similar conclusion holds for z; it follows that for sufficiently large n we have $x=z=h-1$ and the components less than h of the sequence $(m_1,\dots,m_k)$ are 1 and $h-1$ alternately.

If r is even from the inequalities $m_2+m_3\geqslant h,\dots,m_{r-2}+m_{r-1}\geqslant h$, $m_{r+1}+m_{r+2}\geqslant h,\dots,m_{k-2}+m_{k-1}\geqslant h$ we deduce $y\leqslant n-(k-1)h/2+h-2$. Since $m_1=m_k=1$ and $m_i+m_{i+1}\geqslant h$ for every $1\leqslant i\leqslant k-1$ we obtain that $y=n-(k-1)h/2+h-2$ for $x=z=1$.

Now by denoting $x_1=z_1=h-1$, $y_1=n-(k-1)h/2$ and $x_2=z_2=1$ and $y_2=n-(k-1)h/2+h-2$ we get $\lim_{n\to\infty}\varphi(x_1,y_1,z_1)/\varphi(x_2,y_2,z_2)=\infty$.

It follows that for k odd, $k\geqslant 5$, the k-tuple $(m_1,\dots,m_k)$ maximizing g has the property that the single component y greater than $h-1$ is equal to $n-(k-1)h/2$, its order position is odd, its neighboring components are equal to $h-1$ and all components different from y are 1 and $h-1$ alternately.

II. In this case k is even, $k\geqslant 4$; if $y=m_r$ and r is odd, by summing up the inequalities $m_1+m_2\geqslant h,\dots,m_{r-2}+m_{r-1}\geqslant h$, $m_{r+1}+m_{r+2}\geqslant h,\dots,m_{k-2}+m_{k-1}\geqslant h$ one obtains $y\leqslant n-(k-2)h/2-1=n-kh/2+h-1$.

Since $m_1=m_k=1$ and $m_i+m_{i+1}\geqslant h$ for every $1\leqslant i\leqslant k-1$ we deduce that $y=n-kh/2+h-1$ for $z=1$ and $1\leqslant x\leqslant h-1$. In a similar way as for case I we deduce that $x=h-1$ and $z=1$ which imply that all components different from y of the sequence $(m_1,\dots,m_k)$ maximizing g are 1 and $h-1$ alternately.

If r is even then $m_1=1$, $m_2+m_3\geqslant h,\dots,m_{r-2}+m_{r-1}\geqslant h$, $m_{r+1}+m_{r+2}\geqslant h,\dots,m_{k-1}+m_k\geqslant h$ which imply that $y\leqslant n-kh/2+h-1$. We have $y=n-kh/2+h-1$ for $x=1$ and $1\leqslant z\leqslant h-1$ which imply, as above, that $x=1$ and $z=h-1$.

It follows that for k even, $k\geqslant 4$, the k-tuple $(m_1,\dots,m_k)$ maximizing g has the component y greater than $h-1$ equal to $n-kh/2+h-1$, its order position is odd or even and its neighboring components are equal to 1 and $h-1$ so that all components different from y are 1 and $h-1$ alternately.

Now we will deduce the asymptotic expression for $g(n,k,h)$ in three cases: (A) k even, $k\geqslant 4$; (B) k odd, $k\geqslant 5$ and (C) $k=3$.

(A) If k is even, $k\geqslant 4$ from (3) and the characterization of the k-tuples maximizing g as $n\to\infty$ it follows that for sufficiently large n,

$$g(n,k,h)=\frac{(n)_{n-kh/2+h-1}}{((h-1)!)^{(k-2)/2}}2^{\psi(n,k,h)},$$

where

$$\psi(n,k,h) = \binom{n - \frac{kh}{2} + 2h - 2}{h} + \binom{n - \frac{kh}{2} + h - 1}{h - 1}$$

$$= \binom{n}{h} + n^{h-1}\beta(k,h) + \mathrm{O}(n^{h-2})$$

and $\beta(k,h) = (1/2(h-1)!)(4h - kh - 2)$.

(B) For k odd, $k \geqslant 5$ we get that

$$g(n,k,h) = \frac{(n)_{n-(k-1)h/2}}{((h-1)!)^{(k-1)/2}} 2^{\psi(n,k,h)}$$

where for k odd

$$\psi(n,k,h) = 2\binom{n - \frac{(k-1)h}{2} + h - 1}{h} - \binom{n - \frac{(k-1)h}{2}}{h}$$

$$= \binom{n}{h} + n^{h-1}\beta(k,h) + \mathrm{O}(n^{h-2}),$$

and $\beta(k,h) = (1/2(h-1)!)(5h - kh - 4)$.

(C) If $k=3$ from Claim 1 one deduces that the triplet which maximize g is $(1, n-2, 1)$ and $g(n,3,h) = n(n-1)2^{\psi(n,3,h)}$, where

$$\psi(n,3,h) = 2\binom{n-1}{h} - \binom{n-2}{h} = \binom{n}{h} - \binom{n-2}{h-2}$$

$$= \binom{n}{h} - \frac{1}{(h-2)!} n^{h-2}$$

$$+\mathrm{O}(n^{h-3}). \qquad \square$$

Lemma 2.5. *The following equality holds*:

$$\sum_{(n_1,\ldots,n_k) \in D(n,k,h)} g(n,h;n_1,\ldots,n_k) = g(n,k,h)(1 + \mathrm{o}(1))$$

for every $k, h \geqslant 3$.

Proof. If $(m_1,\ldots,m_k) \in D(n,k,h)$ denotes, as above, a k-tuple maximizing $g(n,h;n_1,\ldots,n_k)$ over the domain $D(n,k,h)$ defined by (4), let $(s_1,\ldots,s_k)$ be a k-tuple in $D(n,k,h)$ such that $g(n,h;s_1,\ldots,s_k) = \max\{g(n,h;n_1,\ldots,n_k)$: $(n_1,\ldots,n_k) \in D(n,k,h)$ and $g(n,h;n_1,\ldots,n_k) < g(n,h;m_1,\ldots,m_k)\}$.

From the proof of Theorem 2.4 one can easily deduce that for sufficiently large n we have:

- If k is odd, $k \geqslant 5$, one such system $(s_1,\ldots,s_k)$ has the property that the component greater than $h-1$ is equal to $n-(k-1)h/2-1$, its order position is odd, its neighboring components are equal to h and $h-1$, respectively and all other components are 1 and $h-1$ alternately and they coincide with the components of a k-tuple $(m_1,\ldots,m_k)$ in this case.

- If k is even, $k \geqslant 4$, a system $(s_1, \ldots, s_k)$ has the component greater than $h-1$ equal to $n-kh/2+h-2$, its order position is odd or even and its neighboring components are equal to 1 and h. All other components are equal to 1 and $h-1$ alternately and they coincide with the components of a k-tuple $(m_1, \ldots, m_k)$ maximizing $g(n, h; n_1, \ldots, n_k)$ in this case.
- If $k=3$ we have $(m_1, m_2, m_3)=(1, n-2, 1)$ and $(s_1, s_2, s_3)=(2, n-3, 1)$ or $(1, n-3, 2)$.

Now by straightforward computation one deduces that $g(n, h; m_1, \ldots, m_k)/g(n, h; n_1, \ldots, n_k)$ is greater than $2^{n-kh/2}/n$ for $k \geqslant 4$ and greater than $2^{n-3}/n$ for $k=3$ for every $(n_1, \ldots, n_k) \in D(n, k, h)$ and $(n_1, \ldots, n_k) \notin \{(m_1, \ldots, m_k)$: $(m_1, \ldots, m_k)$ maximizes $g(n, h; n_1, \ldots, n_k)$ in $D(n, k, h)\}$. Since the number of compositions $n = n_1 + \cdots + n_k$ where every $n_i \geqslant 1$ is a natural number equals $\binom{n-1}{k-1}$, a polynomial of degree $k-1$ in n, the proof is complete. $\square$

3. Main results

Theorem 3.1. *We have*

$$2^{\binom{n}{h}-\binom{n-2}{h-2}}(1+o(1)) \leqslant H(n, h; d=2) \leqslant \binom{n}{2} 2^{\binom{n}{h}-\binom{n-2}{h-2}},$$

$$\frac{1}{n-h} g(n, 4, h)(1+o(1)) \leqslant H(n, h; d=3) \leqslant g(n, 4, h)(1+o(1)),$$

$$\frac{1}{2(h-1)} g(n, 5, h)(1+o(1)) \leqslant H(n, h; d=4) \leqslant g(n, 5, h)(1+o(1)),$$

and for every $k \geqslant 5$,

$$H(n, h; d=k) = g(n, k+1, h)(1+o(1)),$$

where $g(n, k, h)$ *is given by Theorem* 2.4.

Proof. The idea of the proof is to start with the lower bound construction by generating a large class of h-hypergraphs of order n and diameter equal to $k \geqslant 2$ and then prove that most h-hypergraphs of diameter k are of this form (upper bound construction).

A. Lower bound. In order to generate a large class of h-hypergraphs of order n and diameter equal to $k \geqslant 2$ we shall proceed as follows:

- If $k=2$ let $u, v \in \{1, \ldots, n\}$ be two distinct vertices and H_1 be an h-hypergraph on vertex set $\{1, \ldots, n\} \setminus \{u, v\}$ such that $d(H_1)=1$. If H consists of H_1 and some edges between u and $V(H_1)$ such that every vertex of H_1 is contained in at least one such an edge and between $V(H_1)$ and v such that v is contained in at least one such an edge, then $d(H)=2$ and H can be chosen in exactly $H(n-2, h; d=1)a(1, n-2)a(n-2, 1)$ ways. By Lemmas 2.1 and 2.3 we have $H(n-2, h; d=1) = 2^{\binom{n-2}{h}}(1+o(1))$ and $a(1, n-2)a(n-2, 1) = 2^{2\binom{n-1}{h}-2\binom{n-2}{h}}(1+o(1))$ which implies that $H(n, h; d=2) \geqslant 2^{\binom{n}{h}-\binom{n-2}{h-2}}(1+o(1))$.

- If $k \geqslant 4$ is even, then $k+1$ is odd and consider a fixed partition π with $k+1$ classes of the set $\{1,\dots,n\}$: $V_1, V_2, \dots, V_{k+1}$ having the following properties:
  (i) $|V_1| = |V_{k+1}| = 1$;
  (ii) $|V_{k-1}| = n - kh/2$;
  (iii) $|V_{k-2}| = |V_k| = h-1$ and all classes different from V_{k-1} contain 1 and $h-1$ elements alternately.

  Let H_1 be any h-hypergraph such that $V(H_1) = V_{k-1}$ and $d(H_1) = 1$. We consider the h-hypergraphs H consisting of H_1, the edges: $V_1 \cup V_2, V_2 \cup V_3, V_3 \cup V_4, \dots, V_{k-3} \cup V_{k-2}$, $V_k \cup V_{k+1}$; some other edges between V_{k-2} and V_{k-1} such that every vertex of V_{k-1} is contained in at least one such an edge and between V_{k-1} and V_k such that every vertex of V_k is included in at least one edge comprising vertices from V_{k-1} and V_k.

  It is clear that each such h-hypergraph H has $d(H) = k$ and the number of these hypergraphs built for a fixed ordered partition π equals $H(n - kh/2, h; d = 1)a(h-1, n-kh/2)a(n-kh/2, h-1) = 2^{\psi(n,k+1,h)}(1 + o(1))$. Now let π be any ordered partition with $k+1$ classes of cardinalities $1, h-1, 1, h-1, \dots, h-1$, $n - kh/2, h-1, 1$ of the set $\{1,\dots,n\}$ and denote by $\mathscr{H}$ the set of all h-hypergraphs of order n and diameter k generated in such a way when π is variable, of the prescribed type.

  If $k = 4$ every hypergraph is generated at most $2(h-1)$ times on this way. Indeed, if π has classes $\{u\}, V_2, V_3, V_4, \{v\}$ and H is an hypergraph generated by π, then the only vertices of degree one of H are u, v and at most $h-2$ vertices of V_2. It follows that H can be eventually generated by the ordered partition with classes $\{v\}, V_4, V_3, V_2, \{u\}$ by considering same edges between V_2 and V_3, V_3 and V_4 and the edges included only in V_3. Also H can be eventually generated by the ordered partition whose classes are $\{w\}, V_2 \cup \{u\} \setminus \{w\}, V_3, V_4, \{v\}$, where w is a vertex of degree one from V_2. Hence $|\mathscr{H}| \geqslant (1/2(h-1))g(n,5,h)(1 + o(1))$ in this case. But if $k \geqslant 6$ every hypergraph H in $\mathscr{H}$ is generated without repetitions because if π has $V_1 = \{u\}$ and $V_{k+1} = \{v\}$ then the only vertices of degree one in H are u, v and at most $h-2$ vertices from V_{k-2}. In this case if $x \in V_{k-2}$ and $d_H(x) = 1$ then $ecc(x) < k = \mathrm{ecc}(u) = \mathrm{ecc}(v)$ and x cannot be interchanged with u or v in the ordered partition which generates H. Also H cannot be generated by the ordered partition π': $\{v\}, V_k, V_{k-1}, \dots, V_2, \{u\}$ with classes of the same cardinalities as π because in the first case the number of vertices at a distance apart from v equal to two equals $n - kh/2$ and in the second one this number equals 1.

  It follows that for even $k \geqslant 6$ the inequalities $H(n,h; d = k) \geqslant |\mathscr{H}| \geqslant g(n, k+1, h)(1 + o(1))$ hold.
- If $k \geqslant 3$ is odd then $k+1$ is even and the construction is somewhat similar to that of the previous case; if $k = 3$ we choose $|V_1| = |V_4| = 1, |V_2| = n-h-1$ and $|V_3| = h-1$ and for $k \geqslant 5$ we define $|V_1| = |V_3| = 1, |V_2| = |V_5| = h-1, |V_4| = n-(k+1)h/2+h-1$ and classes $V_5, V_6, \dots, V_{k+1}$ contain 1 and $h-1$ elements alternately. The number of h-hypergraphs generated by a fixed ordered partition π of the set $\{1,\dots,n\}$ with $k+1$ classes of cardinalities prescribed above is equal to $H(n-(k+1)h/2+h-1, h;$

$d=1)a(1,n-(k+1)h/2+h-1)a(n-(k+1)h/2+h-1,h-1)=2^{\psi(n,k+1,h)}(1+o(1))$ for every odd $k\geqslant 3$. If $k=3$ every hypergraph H in $\mathscr{H}$ is generated with a multiplicity at most equal to $n-h$ since if π has classes $\{u\}, V_2, V_3, \{v\}$ with $|V_3|=h-1$, then v is the unique vertex of degree one in H. It follows that $\{v\}\cup V_3$ is the unique edge E containing v and the ordered partition of $\{1,\dots,n\}\setminus E$ consisting of $\{u\}$ and V_2 can be chosen in $n-h$ ways. Hence in this case $H(n,h;d=3)\geqslant|\mathscr{H}|\geqslant [1/(n-h)]g(n,4,h)(1+o(1))$.

For $k\geqslant 5$ one can show as above that if $V_1=\{u\}$ and $V_{k+1}=\{v\}$ then u and v are the only vertices of degree one in H and the "big" class V_4 is asymmetric relatively to u and v, hence every hypergraph H in $\mathscr{H}$ is generated without repetitions. One obtains

$$H(n,h;d=k)\geqslant\binom{n}{1,h-1,1,n-(k+1)h/2+h-1,h-1,1,\dots,1}$$
$$2^{\psi(n,k+1,h)}(1+o(1))=g(n,k+1,h)(1+o(1)).$$

B. Upper bound. The upper bound for $H(n,h;d=2)$ follows from (1) and the property is proved in this case.

If $v\in V(H)$ has $ecc(v)=k\geqslant 2$, then $\{v\}, V_2(v),\dots,V_{k+1}(v)$ is an ordered partition of $V(H)$, where $V_{i+1}(v)=\{u\colon u\in V(H)$ and $d(u,v)=i\}$ for $1\leqslant i\leqslant k$. It follows that v is adjacent to all vertices of $V_2(v)$ and for every $2\leqslant i\leqslant k$ any vertex $z\in V_{i+1}(v)$ is included in some edge $E\subseteq V_i(v)\cup V_{i+1}(v)$ which contains at least one vertex $t\in V_i(v)$. If we denote $|V_i(v)|=n_i$ for $2\leqslant i\leqslant k+1$ it follows that $n_2\geqslant h-1$ and $n_i+n_{i+1}\geqslant h$ for every $2\leqslant i\leqslant k$. By Lemma 2.2 one gets

$$|\{H\colon V(H)=\{1,\dots,n\}\text{ and }\mathrm{ecc}(v)=k\}|$$
$$=\frac{1}{n}\sum_{(n_1,\dots,n_{k+1})\in D(n,k+1,h)}\binom{n}{n_1,\dots,n_{k+1}}2^{\sum_{i=1}^{k+1}\binom{n_i}{h}}\prod_{i=1}^{k}a(n_i,n_{i+1}).\tag{7}$$

We deduce

$$a(n_i,n_{i+1})\leqslant\alpha(n_i,n_{i+1})=2^{\binom{n_i+n_{i+1}}{h}-\binom{n_i}{h}-\binom{n_{i+1}}{h}}$$

and if $n_i+n_{i+1}=h$ then

$$a(n_i,n_{i+1})=1=2^{\binom{n_i+n_{i+1}}{h}-\binom{n_i}{h}-\binom{n_{i+1}}{h}-1}.$$

Now (7) implies that

$$|\{H\colon V(H)=\{1,\dots,n\}\text{ and }\mathrm{ecc}(v)=k\}|$$
$$\leqslant\frac{1}{n}\sum_{(n_1,\dots,n_{k+1})\in D(n,k+1,h)}g(n,h;n_1,\dots,n_{k+1})=\frac{1}{n}g(n,k+1,h)(1+o(1))$$

by Lemma 2.5. We get $H(n,h;d=k)\leqslant|\bigcup_{v\in\{1,\dots,n\}}\{H\colon V(H)=\{1,\dots,n\}$ and $\mathrm{ecc}(v)=k\}|\leqslant n|\{H\colon V(H)=\{1,\dots,n\}$ and $\mathrm{ecc}(v)=k\}|\leqslant g(n,k+1,h)(1+o(1))$ for every $k\geqslant 3$. The proof is complete. □

Since $g(n,k+1,h) = 2^{\binom{n}{h}+n^{h-1}\beta(k+1,h)+O(n^{h-2})}$, where $\beta(k+1,h) = [1/2(h-1)!](3h-kh-2)$ for every odd $k \geqslant 3$ and $\beta(k+1,h) = [1/2(h-1)!](4h-kh-4)$ for every even $k \geqslant 4$, we obtain:

Corollary 3.2. *For every fixed $h \geqslant 3$ and $k \geqslant 1$ the following equality holds*:

$$\lim_{n\to\infty} \frac{H(n,h;d=k)}{H(n,h;d=k+1)} = \infty.$$

Note that a similar result holds for graphs and digraphs for every fixed $k \geqslant 2$ [3–5].

References

[1] C. Berge, Hypergraphes, Bordas, Paris, 1987.
[2] B. Bollobas, Graph theory, An Introductory Course, Springer, New York, 1979.
[3] D.A. Grable, The diameter of a random graph with bounded diameter, Random Struct. Algorithms 6 (2 and 3) (1995) 193–199.
[4] I. Tomescu, An asymptotic formula for the number of graphs having small diameter, Discrete Math. 156 (1996) 219–228.
[5] I. Tomescu, The number of digraphs with small diameter, Austral. J. Combin. 14 (1996) 221–227.